ON THE VALUATION OF ARITHMETIC-AVERAGE ASIAN OPTIONS: LAGUERRE SERIES AND THETA INTEGRALS

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In a recent significant advance, using Laguerre series, the valuation of Asian options has been reduced in [**D**] to computing the negative moments of Yor's accumulation processes for which functional recursion rules are given. Stressing the role of Theta functions, this paper now solves these recursion rules and expresses these negative moments as linear combinations of certain Theta integrals. Using the Jacobi transformation formula, very rapidly and very stably convergent series for them are derived. In this way a computable series for Black–Scholes price of the Asian option results which is numerically illustrated. Moreover, the Laguerre series approach of [**D**] is made rigorous, and extensions and modifications are discussed. The key for this is the analysis of the integrability and growth properties of the Asia density in [**Y**], basic problems which seem to be addressed here for the first time.

1. Introduction: Asian options are path–dependent options on the arithmetic average of the price of their underlying security. While they are widely traded financial securities their valuation still poses intriguing problems, even in the Black–Scholes setting. The Laguerre series approach of [**D**] so was a significant advance and reduced valuing Asian options to computing the negative moments of the averaging process. For the latter it gave functional recurrence rules. Their structure and significance have been thoroughly analyzed by Yor and his coworkers from a probabilistic point of view.

Taking an analytic point of view, this paper explains how to express these negative moments using Theta functions, and in this way derives computable Laguerre series for the value of the Asian option different from those of [**SE**]. The idea is that using the recurrence relations of [**D**] the negative higher moments of the averaging processes should be linear combinations of certain Theta integrals. This is true in the basic case of §10, and depending on the relative constellation of risk neutral drift and squared volaltility only finitely many correction terms have to be added for the general case of §12. Using the Jacobi transformation formula for Theta series, §11 derives a series for these Theta integrals which is optimized with respect to speed and stability of convergence. We so obtain a Laguerre series for the value of the Asian option whose coefficients are series given by integration against Theta series.

This occurence of Theta functions in valuing the Asian option is rather surprising. Indeed, higher moments of the averaging process have been thoroughly studied in [**YE**, Part A]. In their simplest form they were found to be expressed by Bougerol's identity as the expectation of the respective powers of the hyperbolic sine evaluated on a Brownian motion. While a structural explanation of this link of Asian options with modular is still missing, it highlights a characteristic difficulty of the Laguerre series approach. Its workability crucially depends on the specifics of the option to be valued.

For the Asian option, we think any valid expansion into orthogonal functions will even-

tually be based on Yor's Asia density of [**Y**, §6]. However, it is measurability which is addressed in [**Y**, §6] while for such purposes integrability is required. Thus we establish in §6 integrability of a certain class of functions of exponential type with respect to the Asia density, and put this result to work in deriving two Laguerre series for the value of the Asian option.

First, we illustrate a natural principle for getting series expansions of option prices in terms of higher moments of the option's control variable. The idea is to Laguerre expand in the risk neutral expectation that gives this value the taking–the–non–negative–part function, and then try to put the expectation through the series so obtained. In §7 we combine our integrability results with characteristic mean convergence convergence results to justify this last operation, and so make this series available to valuing Asian options.

Second, there are the Laguerre series in the spirit of [**D**]. Their idea is to construct the price of the Asian option as probability density in the strike price using the notion of ladder height densities, Laguerre expand this density, and try to re–interpret the Laguerre coefficients in terms of the reciprocal of the averaging processes. As explained in §8, Laguerre expandability of the Asian ladder height densities crucially depends on the integrability results of §6, and we have slightly modified the original series in negative moments of the averaging process of [**D**] in addition.

Moreover, we give a second way of re–interpreting the Laguerre coefficients using expectations of gamma function type integrands evaluated at reciprocals of the average process. These expectations also come up naturally if convergence of the Laguerre series in negative moments is studied. Indeed, empirical evidence in §13 suggests that their convergence behaviour is controlled by a certain convergence parameter. Characterizing its optimal values will be in terms of these gamma type expectations. As indicated in §14, the latter now depend on Bessel functions, so have a different character than the negative moments, and we hope to return to their study elsewhere.

2. Preliminaries on Laguerre series: This section collects pertinent properties of Laguerre polynomials from [**L**, §4] and [**S**] fixing any real number $\alpha > -1$. For any non–negative integer n, the *α–Laguerre polynomial* $L_n^\alpha(z)$ is then given by

$$L_n^\alpha(z) = \sum\nolimits_{k=0}^{n} (-1)^k \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} \frac{z^k}{k!(n-k)!} \,,$$

for any complex number z. The first few α–Laguerre polynomials so are $L_0^\alpha(z) = 1$, $L_1^\alpha(z) = 1+\alpha-z$, and $L_2^\alpha(z) = (1/2)\cdot((1+\alpha)(2+\alpha) - 2(2+\alpha)z + z^2)$, and for any positive intger n they satisfy the recurrence relation

$$L_{n+1}^\alpha(z) = \frac{2n+1+\alpha-z}{n+1} L_n^\alpha(z) - \frac{n+\alpha}{n+1} L_{n-1}^\alpha(z) \,.$$

The α–Laguerre polynomials are orthogonal on the positive real line with respect to the weight $w_\alpha(x) = x^\alpha \exp(-x)$ in that $\int_0^\infty w_\alpha(x)(L_n^\alpha(x))^2 L_m^\alpha(x)\,dx = \Gamma(n+\alpha+1)/n!$ and $\int_0^\infty w_\alpha(x) L_n^\alpha(x) L_m^\alpha(x)\,dx = 0$ for any non–negative integers $n \neq m$. To discuss when functions F on the positive real line have a Fourier–type expansion with respect to the α–Laguerre polynomials, consider the case where F is obtained by integrating up a function.

So suppose there is a function f on the positive real line which is integrable on any finite subinterval of the positive real line such that $F(x) = F(0) + \int_0^x f(y)\, dy$, for any $x > 0$. Moreover assume that the functions $\sqrt{w_\alpha}\, F$ and $\sqrt{w_{\alpha+1}}\, f$ are square integrable on $(0, \infty)$. Then F is represented by the α–*Laguerre series*

$$e^{-\frac{x}{2}} x^{-\frac{x}{2}} F(x) = \sum_{n=0}^{\infty} c_n e^{-\frac{x}{2}} x^{-\frac{x}{2}} L_n^\alpha(x)$$

which converges absolutely and uniformly for x in any subinterval $[c, \infty)$ of the positive real line. Any n–th generalized Fourier coefficient c_n of this series is given by

$$c_n = \frac{n!}{\Gamma(n+\alpha+1)} \int_0^\infty e^{-x} x^\alpha F(x) L_n^\alpha(x)\, dx\,.$$

Alternatively, if F is continuous and $\sqrt{w_\alpha}\, F$ is square integrable on $(0, \infty)$, then F is also represented by the above Laguerre series, see [**L**, §4.23]. Convergence at zero of this series, however, needs a separate study, see for instance [**S**, p.366f].

3. Preliminaries on Theta functions: Following [**M**], the concept to be discussed is the classical Riemann *Theta function* ϑ given for any complex number z and any positive real number t by:

$$\vartheta(z|\, t) = \frac{1}{\sqrt{\pi t}} \sum\nolimits_{n \in \mathbf{Z}} \; e^{-\,(z+n)^2 \cdot \frac{1}{t}}\,.$$

This series converges absolutely, and uniformly on compact sets. Thus ϑ can be seen as a holomorphic function on the product of the complex plane with the upper complex half–plane. In modular forms this is usually done such that the above Theta function would be considered as evaluated not at t but at the point $i\pi t$ of the upper half–plane. Theta functions are a basic class of holomorphic modular forms. They have been studied since the eighteenth century and are interrelated with a number of areas of central importance for mathematics, as number theory, algebraic geometry, classical analysis, and partial differential equations.

Two such interrelations are to be described. For the first recall that $t^{-1/2} \exp(-\pi x^2/t)$ is the fundamental solution of the Heat equation on the line with initial data at $t = 0$ a delta function at $x = 0$. Thus ϑ at real arguments can be seen as the superposition of infinitely many such solutions with initial data being delta functions at the half–integers $x = n + 1/2$.

From the point of view of modular forms, Theta functions can be characterized by a certain periodicity behaviour with respect to each of their two variables. Here, the behaviour with respect to the second variable is deeper and more subtle. It is expressed by a functional equation with respect to the second variable, the *Jacobi transformation formula*. Restricting to real arguments, this formula is the following remarkable identity:

$$1 + 2 \sum\nolimits_{n=1}^{\infty} (-1)^n e^{-(\pi n)^2 \cdot t} \cos(2\pi n x) = \frac{1}{\sqrt{\pi t}} \sum\nolimits_{n \in \mathbf{Z}} \; e^{-\,(z - \frac{1}{2} + n)^2 \cdot \frac{1}{t}}\,,$$

for any real numbers x and $t > 0$. The left hand side of this identity, the Jacobi transform of ϑ, is rapidly converging for t large, whereas its right hand side is rapidly converging for t small. The Jacobi Transformation Formula is proved by Fourier analysis, exhibiting ϑ as the Fourier expansion with respect to the second variable of its Jacobi transform.

4. Basic notions: First we discuss the notions basic for the analysis of Asian options. We work in the Black–Scholes framework using the risk–neutral approach to the valuation of contingent claims. In this set–up there are two securities. There is a riskless security, a bond, whose price grows at the continuously compounding positive interest rate r. There is also a risky security, whose price process S is modelled as follows. Consider a complete probability space equipped with the standard filtration of a standard Brownian motion on the time set $[0,\infty)$. On this filtered space, we have the risk neutral measure Q, which is a probability measure equivalent to the given one. And then we have a standard Q–Brownian motion B such that S is the strong solution of the following stochastic differential equation:

$$dS_t = \varpi \cdot S_t \cdot dt + \sigma \cdot S_t \cdot dB_t, \qquad t \in [0,\infty).$$

The positive constant σ is the volatility of S. The specific form of the otherwise arbitrary constant ϖ depends on the nature of the security modelled (eg. stock, currency, commodity etc.). For example it is the interest rate if S is a non–dividend–paying stock.

Fix any time t_0 and consider the accumulation process J given for any time t by:

$$J(t) = \int_{t_0}^{t} S_u \, du \, .$$

The European–style *arithmetic–average Asian option* written at time t_0, with maturity T, and fixed–strike price K is then the contingent claim on the closed time interval from t_0 to T paying $(J(T)/(T-t_0)-K)^+ := \max\{0, J(T)/(T-t_0)-K\}$ at time T. Recall that points in time are taken to be non–negative real numbers. The price C_t of the Asian option at any time t between t_0 and T is given as the following risk neutral expectation

$$C_t = e^{-r(T-t)} E^Q \left[\left(\frac{J(T)}{T-t_0} - K \right)^+ \Big| \mathcal{F}_t \right]$$

which is conditional on the information $\mathcal{F}_t$ available at time t. However, following [**GY**, §3.2], do not focus on this price. As described there in great detail, we instead normalize the valuation problem, consider the factorization:

$$C_t = \frac{e^{-r(T-t)}}{T-t_0} \cdot \frac{4S_t}{\sigma^2} \cdot C^{(\nu)}(h,q) \, ,$$

and so reduce to computing

$$C^{(\nu)}(h,q) = E^Q \big[(A_h^{(\nu)} - q) \big],$$

the *normalized time–t price* of the Asian option. To explain the notation, $A^{(\nu)}$ is Yor's twohundred percent volatility accumulation process

$$A_h^{(\nu)} = \int_0^h e^{2(B_w + \nu w)} dw \, ,$$

and the normalized parameters are as follows:

$$\nu = \frac{2\varpi}{\sigma^2} - 1, \qquad h = \frac{\sigma^2}{4}(T-t), \qquad q = kh + q^*,$$

where

$$k = \frac{K}{S_t}, \qquad q^* = q^*(t) = \frac{\sigma^2}{4S_t}\left(K\cdot(t-t_0) - \int_{t_0}^{t} S_u\,du\right).$$

To interpret these quantities, ν is the *normalized adjusted interest rate*, h is the *normalized time to maturity*, which is non–negative, and q is the *normalized strike price*.

5. A reduction of the valuation problem: Computing the normalized time–t price of the Asian option reduces to the case where the normalized strike price q is positive. Indeed, if q is non–positive, Asian options loose their option feature, and their normalized time–t price is given by

$$C^{(\nu)} := E^Q\left[\left(A_h^{(\nu)} - q\right)^+\right] = E^Q\left[A_h^{(\nu)}\right] - q\,.$$

On applying Fubini's theorem, this last expectation is computed as follows:

$$E^Q\left[A_h^{(\nu)}\right] = \frac{e^{2h(\nu+1)} - 1}{2(\nu+1)}\,,$$

for ν any real number. The right hand side is analytic in ν with its value at $\nu = -1$ being equal to h. For computing $C^{(\nu)}$ if q is positive we discuss two Laguerre series. They in turn crucially depend on an extension of a basic result of Yor's about the Asia density to be discussed next.

6. Yor's Asian option density revisited: This section discusses growth and integrability properties of the density of the accumulation processes $A^{(\nu)}$. This is based on Yor's integral representations of [**Y**, (6.e), p.528] recalled below and extends his measurability results. Recall the Girsanov identity for the normalized time–t price

$$C^{(\nu)} = E^Q\left[(A_h^{(\nu)} - q)^+\right] = e^{-\frac{\nu^2 h}{2}} E\left[(A_h^{(0)} - q)^+ e^{\nu W_h}\right]$$

which follows on making $W_t = \nu t + B_t$ a Brownian motion and dropping reference to the resulting measure. It reduces computing $C^{(\nu)}$ to a corresponding problem in the zero drift accumulation process $A^{(0)}$ at the cost of introducing a second stochastic factor in the integrand. Define the density $\alpha_{\nu,h}$ on the positive real line by

$$\alpha_{\nu,h}(x) = e^{-\frac{x^2}{2}} \int_0^\infty g_\nu(y) e^{-\frac{xy^2}{2}} \psi_{xy}(h)\,dy\,,$$

for any $x > 0$. Here g_ν is the map on the positive real line that sends any $y > 0$ to its ν–th power, and $\psi_\xi(h)$ is for any positive real number ξ given by:

$$\psi_\xi(h) = \int_0^\infty e^{-\frac{w^2}{2h} - \xi\cosh(w)} \sinh(w) \sin\left(\frac{\pi w}{h}\right) dw\,.$$

To address integrability let f be any continuous function on the positive real line, and for any real numbers a, b define the weighted maps $f_{a,b}$ by $f_{a,b}(x) = \exp(x^{-1}a)x^b f(x)$, for any positive real number x. Then we have the

Proposition: *If $a < 1/2$ and f behaves like a power map near the origin and towards infinity, the map sending any positive real number x to $f(x^{-1})\exp(ax)x^{-b}\alpha_{\nu,h}(x)$ extends to an integrable map on the non–negative real line, and we have the identity*

$$E\big[f_{a,b}(A_h^{(0)})g_\nu(e^{W_h})\big] = c_h \int_0^\infty f_{a,b}\Big(\frac{1}{x}\Big)\alpha_{\nu,h}(x)\,dx$$

of finite integrals, where $c_h = (2\pi^2 h)^{-1/2}\exp((2h)^{-1}\pi^2)$. Moreover, the map sending any positive real number x to $\exp(ax)x^{-b}\alpha_{\nu,h}(x)$ extends to an integrable and bounded map on the non–negative real line.

Remark: Examples of functions f which meet the conditions of the Proposition are furnished by any rational function with no poles on the positive real line, and by any function sending $x \geq 0$ to $(x-q)^+$ or $(q-x)^+$.

The proof of the Proposition is based on Yor's triple integral [**Y**, (6.e), p.528] and his observation [**Y**, (6.g), p.529]. Recall the latter implies that for any non–negative integer k the map $\xi \mapsto \psi_\xi(h)$ is of order big Oh of ξ^k with going ξ to zero, while the former asserts

$$E\big[h(A_h^{(0)})g(e^{W_h})\big] = c_h \int_0^\infty \int_0^\infty h\Big(\frac{1}{x}\Big)e^{-\frac{x}{2}}g(y)e^{-\frac{xy^2}{2}}\psi_{xy}(h)\,dx\,dy$$

for any Borel measurable functions g, h on the non–negative real line into itself. In the Yor triple integral for the Proposition you would like to interchange the order of integration. With the functions f and $\xi \mapsto \psi_\xi(h)$ possibly negative, Tonelli's theorem does not apply for this. However it does apply on taking absolute values of the integrands. And if the resulting integral can be proved finite this will give integrability and justify interchanging the order of integration in the original Yor's triple integral using Fubini's theorem. Abbreviating $\gamma = \nu - b$ and $\delta = 1/2 - a$ we are so reduced to prove finiteness of

$$J = \int_0^\infty |f|\Big(\frac{1}{x}\Big)e^{-\delta x}D_{0,\infty}(x)\,dx\,,$$

setting

$$D_{A,B}(x) = \int_A^B \frac{e^{-\frac{\xi^2}{2x}}}{x^{\gamma+1}}\cdot\xi^\gamma\cdot|\psi_\xi(h)|\,d\xi\,.$$

Decompose J into four parts by breaking the two integrations at the point 1. The idea is to first majorize the respective inner integrals $D_{A,B}$ using Yor's above observation, and then use the exponential decay of the integrand towards infinity to bound the whole summand. We discuss this for the case where x and ξ both range from 1 to infinity. Fixing $x > 1$ and changing variables $\eta = (2x)^{-1/2}\eta$, we have

$$D_{1,\infty}(x) = \frac{\sqrt{2x}}{x^{\gamma+1}}\int_{1/\sqrt{2x}}^\infty e^{-\eta^2}\cdot(\eta\sqrt{2x})^c\cdot|\psi_{\eta\sqrt{2x}}(h)|\,d\eta.$$

To majorize this last integral notice that $(2x)^{1/2}\eta$ is greater than or equal to 1 by construction. On $[1,\infty)$ the map $y \mapsto y^\gamma\exp(-y)$ is bounded by a constant A_γ depending

only on γ. Use this in the defining integral for ψ to obtain

$$y^\gamma|\psi_y(h)| \leq \int_0^\infty e^{-\frac{w^2}{2h}} y^\gamma e^{-y} \sinh(w)\, dw \leq A_\gamma \int_0^\infty e^{-\frac{w^2}{2h}} \sinh(w)\, dw\,.$$

Thus there is a constant $B_{\gamma,h}$ depending only on γ and h such that $D_{1,\infty}(x)$ sits below $B_{\gamma,h} \cdot x^{-(\gamma+1/2)}$ for any x in $[1,\infty)$. For the respective summand of J we so obtain,

$$\int_1^\infty |f|\Big(\frac{1}{x}\Big) e^{-\delta x} D_{1,\infty}(x)\, dx \leq B_{\gamma,h} \int_1^\infty |f|\Big(\frac{1}{x}\Big) \cdot \frac{e^{-\frac{\delta}{2}x}}{x^{\gamma+1/2}} \cdot e^{-\frac{\delta}{2}x}\, dx\,.$$

Now apply the above reasoning to the middle factor of the last integral's integrand and bound it by $(\delta/2)^{\gamma+1/2} A_{-(\gamma+1/2)}$. On substitution,

$$\int_1^\infty |f|\Big(\frac{1}{x}\Big) e^{-\delta x} D_{1,\infty}(x)\, dx \qquad\qquad \leq B_{\gamma,h} \cdot \Big(\frac{\delta}{2}\Big)^{\gamma+1/2} A_{-(\gamma+1/2)} \int_1^\infty |f|\Big(\frac{1}{x}\Big) \cdot e^{-\frac{\delta}{2}x}\, dx\,,$$

and this last integral is finite since δ is positive by construction and f behaves like a power map near the origin by hypothesis. The remaining three summands of J can be majorized in a similar way, and this completes the proof.

7. First Laguerre series for the Asian option: As a first application of our study of the growth behaviour of Yor's Asia density we illustrate a general priciple for getting series expansions of option prices in terms of higher moments of the option's control variable. The idea is to Laguerre expand in the risk neutral expectation that gives this value the taking–the–non–negative–part function, and then try to put the expectation through the series so obtained. The krux of this *Ansatz* is to justify the last operation, and if such a justification can be found it will be specific to the option considered. For the Asian option we have been able to give such a justification using the properties of the Asia density established in §6 Proposition in an essential way. The precise result is the following series in terms of the moments of the reciprocal of Yor's accumulation process.

Lemma: *Let c be any positive real number with $0 < qc < 1/2$. Setting $X = A_h^{(\nu)}$, the normalized time–t price $C^{(\nu)}$ of the Asian option has the absolutely convergent series*

$$C^{(\nu)} = \frac{1}{c} \sum_{n=0}^\infty c_n \cdot E^Q\Big[X \cdot L_n^\alpha\Big(\frac{qc}{X}\Big)\Big]$$

whose any n–th coefficient c_n is given by

$$c_n = \sum_{k=0}^n \frac{(-1)^{k+1}}{\Gamma(k+\alpha+1)} \binom{n}{k} \Big((k+\alpha) \cdot \gamma(k+\alpha+1, c) + c^{k+\alpha+1} e^{-c}\Big),$$

with $\gamma(s,a) = \int_0^a \exp(-x) x^{s-1}\, dx$ for any a, $s > 0$ the incomplete gamma function, and

$$E^Q\Big[X \cdot L_n^\alpha\Big(\frac{qc}{X}\Big)\Big] = \sum_{k=0}^n (-1)^k \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} \frac{(qc)^k}{k!(n-k)!} E^Q\Big[\frac{1}{X^{k-1}}\Big]\,.$$

Remark: In [**D**, Theorem 5.1, p.417f] a Laguerre series for the the density of the reciprocal of Yor's accumulation process is given. Use it to compute the expectation defining $C^{(\nu)}$ by a formal term by term integration. The series so obtained is then seen to be identical with that of the Lemma.

The proof of the Lemma illustrates the basic principles for working with orthogonal polynomials. Take any function ϕ on the positive real line which has an α–Laguerre series, so let $w^{1/2}\phi$ be square integrable where w is the weight $w(x) = \exp(-x)x^\alpha$. From [**AAR**, Theorem 6.5.3, p.308] we have mean convergence with respect to w of this series to ϕ. Writing c_k for any k–th Laguerre coefficient and $R_n = \phi - (c_0 L_0^\alpha + \cdots + c_n L_n^\alpha)$ for any n–th order remainder term of the Laguerre series, this is equivalent to

$$\lim_{n\to\infty} \int_0^\infty w(x) \cdot [R_n(x)]^2 \, dx = 0\,.$$

The idea for the proving the Lemma is to transcribe the ordinary convergence of the series to be proved into mean convergence of the Laguerre series for $\phi_c(x) = (c–x)^+$ and apply this mean convergence result. Thus let R_n now denote any n–th order remainder term of the Laguerre series for ϕ_c and set $\rho_n(x) = xR_n(x^{-1}qc)$. Observing $(x-q)^+ = c^{-1}x\phi_c(x^{-1}qc)$,

$$E^Q\Big[\big(X-q\big)^+\Big] - \frac{1}{c}\sum\nolimits_{k=0}^{n} c_k E^Q\Big[X \cdot L_k^\alpha\Big(\frac{qc}{X}\Big)\Big] = \frac{1}{c} E^Q\Big[\rho_n\big(X\big)\Big]\,.$$

The proof of the Lemma so reduces to show that this last expectation goes to zero with n to infinity. Using the Asian density $\alpha_{\nu,h}$ of §6 Proposition, we have

$$E^Q\Big[\rho_n\big(X\big)\Big] \le c_0 \int_0^\infty \alpha_{\nu,h}(x)\rho_n\Big(\frac{1}{x}\Big)\, dx\,,$$

with $c_0 = c_h \exp(-\nu^2 h/2)$. Applying Cauchy–Schwarz,

$$\begin{aligned}\Big|E^Q\Big[\rho_n\big(X\big)\Big]\Big|^2 &\le c_0^2 \int_0^\infty \frac{1}{x^2}\alpha_{\nu,h}(x)\, dx \cdot \int_0^\infty |R_n(qcx)|^2 \alpha_{\nu,h}(x)\, dx \\ &\le \frac{c_0}{qc} E^Q\big[X^2\big] \cdot \int_0^\infty \big|R_n(\xi)\big|^2 \alpha_{\nu,h}\Big(\frac{\xi}{qc}\Big)\, d\xi\,,\end{aligned}$$

on changing variables $\xi = qcx$ for the last equality. Since all higher moments of any Yor's accumulation process are finite using [**Y**, §4], we are further reduced to show that the second integral factor goes to zero with n to infinity. This is where the growth properties of the Asian option density enter in an essential way. Indeed, using the hypothesis $0 < qc < 1/2$ choose any $qc < \beta < 1/2$. Then §6 Proposition applies to show that $x \mapsto \exp(\beta x)x^{-\alpha}\alpha_{\nu,h}(x)$ is bounded on the positive real line, whence

$$\xi \mapsto e^\xi \cdot \Big(\frac{\xi}{qc}\Big)^{-\alpha} \cdot \alpha_{\nu,h}\Big(\frac{\xi}{qc}\Big)$$

is bounded on the non–negative real line as well. As a consequence there is a positive constant D such that

$$\int_0^\infty |R_n(\xi)|^2 \alpha_{\nu,h}\Big(\frac{\xi}{qc}\Big)\, d\xi \le D \int_0^\infty e^{-\xi}\xi^\alpha |R_n(\xi)|^2 \, d\xi = D \int_0^\infty w(\xi)|R_n(\xi)|^2 \, d\xi\,.$$

Now the mean convergence result recalled at the beginning applies to give that this last integral goes to zero with n to infinity. Absolute convergence of the series follows using Riemann's criterion for conditional convergence: with the above series also any rearrangement of it converges to the same function. This completes the proof of the Lemma.

8. Ladder height density Laguerre series for the Asian option: A second Laguerre expansion of the value of the Asian option has been proposed in [**D**]. The idea is to construct this value as a probability density in the strike price, Laguerre expand, and try to re–interpret the Laguerre coefficients in terms of the reciprocal of Yor's accumulation process. The key notion is that of the *ladder height density* $\overline{g}_Z$ associated to a probability density function g_Z of any positive Q–integrable random variable Z. It is defined for any $c > 0$ by

$$\overline{g}_Z(c) = \frac{1}{E^Q[Z]} \int_c^\infty w \cdot g_Z(w)\, dw\,.$$

We have slightly modified the argument of [**D**, §7] how this applies to Asian options. Using §5 assume the modified strike price q positive. For any positive real number c, define the Q–probability density function g of a positive and Q–integrable random variable by

$$g(y) = \frac{1}{E[Y^{-1}]} \cdot \frac{f_Y(y)}{y} \qquad \text{where} \qquad Y = \frac{qc}{A_h^{(\nu)}}\,.$$

Then we have the representation of the normalized time–t price $C^{(\nu)}$ of the Asian option in terms of the ladder height density $\overline{\overline{g}}$ associated to $\overline{g}$

$$C^{(\nu)} = E^Q\big[X\big] - q + \frac{q^2 c}{2} E^Q\big[X^{-1}\big] \cdot \overline{\overline{g}}(c)$$

setting $X = A_h^{(\nu)}$. Postponing the computation of $E[X^{-1}]$ to §12 Lemma and with $E^Q[X]$ computed in §5, we are so reduced to computing $\overline{\overline{g}}(c)$. The idea is to try to use a Laguerre series for this. Thus consider for any positive real number c and any real numbers β and δ the formal Laguerre series expansion

$$c^\beta e^{-\delta c} \cdot \overline{\overline{g}}(c) = \sum\nolimits_{n=0}^{\infty} c_n L_n^\alpha(c)$$

whose any n–th Laguerre series coefficient is given by

$$c_n = \sum\nolimits_{k=0}^{n} \frac{(-1)^k}{\Gamma(k+\alpha+1)} \binom{n}{k} \cdot I_{\beta,\delta,k}\,.$$

where $I_{\beta,\delta,k}$ abbreviates the integral

$$I_{\beta,\delta,k} = \int_0^\infty y^{\alpha+\beta+k} e^{-(1+\delta)y} \cdot \overline{\overline{g}}(y)\, dy\,.$$

The basic question is under which conditions on β and δ this series converges, and this is again crucially based on the analysis of Yor's density in §6. The second question is how to compute the coefficient integrals $I_{\beta,\delta,k}$ for these admissible values of β and δ. Here is the following extension of [**D**, Theorem 7.1, p.422].

Proposition: *Assume $q > 0$, abbreviate $X = A_h^{(\nu)}$, and let β and δ be such that $\alpha+2\beta > -1$ and $2\delta > -(1+(2qc)^{-1})$. Then the above α–Laguerre series converges. If moreover $qc < 1/2$, we can choose $\delta = -1$ as an admissible such value, and*

$$I_{\beta,-1,k} = \frac{2}{E^Q[X^{-1}]} \cdot \frac{(qc)^{\alpha+\beta+k}}{(\alpha+\beta+k+1)(\alpha+\beta+k+2)} E^Q\Big[\frac{1}{X^{\alpha+\beta+k+1}}\Big],$$

for any non–negative integer k. If $\delta \neq -1$ and $m = \alpha+\beta+k$ is any non–negative integer,

$$I_{\beta,\delta,k} = \frac{2qc}{E^Q[X^{-1}]} \frac{m!}{(1+\delta)^{m+1}} - \frac{2}{E^Q[X^{-1}]} \sum_{\ell=0}^{m} \frac{m!}{(1+\delta)^{\ell+1}} \Big\{ \frac{E[X^{-1}]}{(1+\delta)^{m-\ell}} - \sum_{p=0}^{m-\ell} \frac{(qc)^{m-\ell-p}}{(m-\ell-p)!(1+\delta)^{p+1}} E\Big[\frac{1}{X^{m-\ell-p-1}} e^{-\frac{qc(1+\delta)}{X}}\Big]\Big\}.$$

To prove the Proposition, given [**D**, Theorem 5.1, p.418] we first have to check square integrability, i.e., have to determine which maps $x \mapsto x^{\alpha+2\beta} \exp(-(1+2\delta)x)(\overline{\overline{g}}(x))^2$ are integrable on $(0,\infty)$. To get a hold on $\overline{\overline{g}}$, recall if $X_{\overline{g}}$ is any random variable with Q–density function $\overline{g}$, we have $E^Q[X_{\overline{g}}]\overline{\overline{g}}(x) = Q(\exp(aX_{\overline{g}}) > \exp(ax))$ for any $a > 0$. Hence Markov's inequality gives $E^Q[X_{\overline{g}}]\overline{\overline{g}}(x) \leq \exp(-ax)E^Q[\exp(aX_{\overline{g}})]$. On substitution we are so reduced to characterize finiteness of

$$\frac{E^Q[\exp(aX_{\overline{g}})]}{E^Q[X_{\overline{g}}]} \int_0^\infty x^{\alpha+2\beta} e^{-(1+2(\delta+a))x} dx\,.$$

The integral factor gives the conditions $\alpha + 2\beta > -1$ and $1 + 2(a+\delta) > 0$. Since $E^Q[X_{\overline{g}}] = E^Q[X^2]/2$ using [**D**, Theorem 2.1, p.411] and all moments of X are finite, we have to determine those $a > 0$ for which $E^Q[\exp(aX_{\overline{g}})]$ is finite. Since on unravelling definitions

$$E^Q[\exp(aX_{\overline{g}})] = \frac{E^Q[X]}{aqc}\Big(\frac{1}{E^Q[X]} E^Q\Big[X e^{\frac{aqc}{X}}\Big] - 1\Big),$$

this again is an application of §6 Proposition. The expectation will be finite for any positive real number a such that $aqc < 1/2$ yielding $2\delta > -(1+(2qc)^{-1})$. This is satisfied for $\delta = -1$ if $qc < 1/2$, and in this case the coefficient integrals $I_{\beta,\delta,k}$ have been computed in [**D**, Theorem 2.1, p.411]. An analogous computation gives the coefficient integrals in the other case which is not contained in [**D**]. This completes the proof of the Proposition.

9. Review of two results of Dufresne: As one of his insights into the structure of the accumulation process

$$A_h^{(\nu)} = \int_0^h e^{2(\nu w + B_w)} dw\,,$$

Yor describes in [**Y**, (4.k), p.522] how its higher moments are determined by certain Gauss hypergeometric functions ${}_2F_1$. These results have been extended in [**D**, §4] to the higher

moments of the reciprocals of these accumulation processes in the following way. First we have the integral representation

$$E^Q\Big[\frac{1}{A_h^{(\nu)}}\Big] = 2\frac{e^{-\frac{\nu^2 h}{2}}}{\sqrt{2\pi h^3}}\int_0^\infty y e^{-\frac{y^2}{2h}}\frac{\cosh((\nu-1)y)}{\sinh(y)}\,dy\,,$$

for any real number ν. At the base of this result is Yor's description of the Laplace transform with respect to time of the first moment of $A^{(\nu)}$ using the confluent hypergeometric function Φ. This description remains valid *mutatis mutandis* for the expectation of r–th powers of $A^{(\nu)}$ where $r > -1$. The identity resulting on inversion is in terms of the Gauss hypergeometric function and holds for $r > -3/2$ using analytic continuation.

The higher moments of the reciprocal of $A^{(\nu)}$ are determined recursively from its first moment. Considering any k–th such moment

$$m_k(h) = E^Q\big[\big(A_h^{(\nu)}\big)^{-k}\big]$$

as a function in the time variable $h > 0$, we have

$$m_k = 2(k-(\nu{+}1))m_{k-1} - \frac{1}{k-1}m'_{k-1}\,,$$

for any $k \geq 2$. This is proved using the Itô Lemma on applying time reversal. Different angles on this are discussed in [**Y1**], [**Y2**], and [**Y3**] in particular.

10. Higher moments as Theta integrals in the basic case: It is Theta functions as discussed in §3 which provide the key for understanding the higher moments of the reciprocal of Yor's accumulation process. With higher moments of Yor's accumulation processes themselves being determined by hyperbolic sines [**Y**,4.g, p.519], this appearance of theta functions seems unexpected at least. This section considers the *basic case* $|\nu-1| \leq 1$ in which the higher moments of the reciprocal of any time–h value of $A^{(\nu)}$

$$m_k(h) = E^Q\big[\big(A_h^{(\nu)}\big)^{-k}\big]$$

are completely determined by theta functions. Indeed, computing them reduces to computing the Theta integrals Θ_k given by

$$\Theta_k(h) = \frac{1}{2\sqrt{2}}\int_0^\infty \vartheta\Big(\frac{\nu}{2}\Big|w\Big)\frac{w^{k-1}}{(wh+1/2)^{k+1/2}}\,dw\,,$$

for any $h > 0$ and for any positive integer k. The precise result is the following

Lemma: *Suppose $|\nu-1| \leq 1$. For any positive integer n, we have*

$$e^{\frac{\nu^2 h}{2}}m_n(h) = a_{n,1}\Theta_1(h) + \cdots + a_{n,n}\Theta_n(h)\,,$$

for any $h > 0$, where the coefficients $a_{n,k}$ are recursively determined by $a_{1,1} = 1$ and the recurrence relations

$$\begin{aligned} a_{n+1,1} &= \Big(2(n-\nu) + \frac{\nu^2}{2n}\Big) a_{n,1}\,, \\ a_{n+1,n+1} &= \Big(1 + \frac{1}{2n}\Big) a_{n,n}\,, \\ a_{n+1,k} &= \Big(2(n-\nu) + \frac{\nu^2}{2n}\Big) a_{n,k} + \Big(\frac{k-1}{n} + \frac{1}{2n}\Big) a_{n,k-1}\,, \end{aligned}$$

for any k between 2 and n.

Remark: Similar but different charactarizations of these higher moments have been obtained by Yor and coworkers if $\nu = 0$, see in particular [**Y1**], [**Y2**], and [**Y3**, §4].

The key insight for the proof is that in Dufresne's integral for the expectation recalled in §9 the hyperbolic quotient factor of the integrand is the Laplace transform at y^2 of a theta function if $|\nu - 1| \le 1$. Indeed, for any positive real number y we then have

$$\frac{\cosh((\nu-1)y)}{y \sinh(y)} = \int_0^\infty e^{-y^2 w} \vartheta\Big(\frac{\nu}{2}\Big| w\Big)\, dw\,.$$

Applying Fubini's theorem the expression of the Lemma for $m_1(h)$ follows from this. An induction using the recursion relation for m_k of §9 then completes the proof.

11. Computing the Theta integrals: This section discusses two series for computing the Theta integrals Θ_n. From §10 these are recalled to be given by

$$\Theta_n(h) = \frac{1}{2\sqrt{2}} \int_0^\infty \vartheta\Big(\frac{\nu}{2}\Big| w\Big) \frac{w^{n-1}}{(wh + 1/2)^{n+1/2}}\, dw\,,$$

for any $h > 0$ and for any positive integer n.

First series: First we have the following absolutely convergent series in terms of the confluent hypergeometric function of the second kind Ψ

$$\Theta_n(h) = \frac{\Gamma(n)}{2h^n} \Psi\Big(n, \frac{1}{2}; 0\Big) + \frac{\Gamma(n)}{h^n} \sum\nolimits_{m=1}^\infty \cos(\pi m \nu) \cdot \Psi\Big(n, \frac{1}{2}; \frac{(\pi m)^2}{2h}\Big)\,.$$

It is obtained by term by term integration of the series which results from applying the Jacobi transformation formula of §3 to the Theta function factor in $\Theta_n(h)$. However, it is problematic in that confluent hypergeometric functions are most complicated functions and their implementations are often flawed. Similar series for $\nu = 0$ are in [**Y3**, §4].

Second series: Still there is a second series in terms of less problematic functions. It is the absolutely convergent series

$$\Theta_n(h) = \sum_{m=0}^\infty c_{B,m}(h) + \sum_{m \in \mathbf{Z}} d_{B,m}(h)$$

whose coefficients are for any positive real number B given as follows:

$$c_{B,0}(h) = \frac{1}{2h^n}\sum_{k=0}^{n-1}\frac{(-1)^{n-1-k}}{n-k-\frac{1}{2}}\binom{n-1}{k}\frac{1}{(2hB+1)^{n-k-1/2}},$$

$$c_{B,m}(h) = \gamma_m\sum_{k=0}^{n-1}(-2)^k\binom{n-1}{k}\frac{h^k}{(\pi m)^{2k}}\left[C^{(1)}_{m,k}+C^{(2)}_{m,k}\right]$$

for any positive intgeger m, abbreviating

$$\gamma_m = \frac{\cos(\pi m\nu)}{\sqrt{2}}\frac{(-1)^{n-1}}{2^{n-1}}\frac{(\pi m)^{2n-1}}{h^{2n-1/2}},$$

$$C^{(1)}_{m,k} = (-1)^{n-k-1}\sum_{\ell=0}^{n-k-1}(-1)^\ell\frac{\left(\frac{1}{2}\right)_\ell}{\left(\frac{1}{2}\right)_{n-k}}\cdot\frac{e^{-(\pi m)^2B}}{\left((\pi m)^2\left(B+\frac{1}{2h}\right)\right)^{\ell+1/2}},$$

$$C^{(2)}_{m,k} = \sqrt{\pi}\,\frac{(-1)^{n-k}}{\left(\frac{1}{2}\right)_{n-k}}\cdot e^{-(\pi m)^2B}\,W\left(\pi m\sqrt{B+(2h)^{-1}}\right).$$

If m is any integer different from $-\nu/2$, we have

$$d_{B,m}(h) = \frac{(-2)^{n-1}}{\sqrt{\pi}}\sum_{k=0}^{n-1}D^{(1)}_{m,k}+D^{(2)}_m$$

abbreviating

$$D^{(1)}_{m,k} = (-1)^k\frac{\left(\frac{1}{2}\right)_k}{\left(\frac{1}{2}\right)_n}\cdot\frac{\left(m+\frac{\nu}{2}\right)^{2(n-k-1)}}{\left(\frac{1}{B}+2h\right)^{k+1/2}}\cdot e^{-\frac{1}{B}\left(m+\frac{\nu}{2}\right)^2}$$

$$D^{(2)}_m = 2^{n-1}\frac{(-1)^n}{\left(\frac{1}{2}\right)_n}\left|m+\frac{\nu}{2}\right|^{2n-1}e^{-\frac{1}{B}\left(m+\frac{\nu}{2}\right)^2}\,W\left(\left|m+\frac{\nu}{2}\right|\sqrt{B^{-1}+2h}\right),$$

while for $m=-\nu/2$ any integer, we have

$$d_{B,-\frac{\nu}{2}} = \frac{2^{n-1}}{\sqrt{\pi}\left(n-\frac{1}{2}\right)}\cdot\frac{B^{n-1/2}}{(2hB+1)^{n-1/2}}.$$

Here the function W is given by $W(z) = \exp(z^2)\mathrm{Erfc}(z)$, for any complex number z, and $(\lambda)_k$ is the Pochhammer symbol of any complex number λ recursively given by $(\lambda)_0 = 1$ and $(\lambda)_{k+1} = (\lambda-k)\cdot(\lambda)_k$ for any non–negative integer k. The idea for this series is to optimize the convergence behaviour using the Jacobi transformation formula of §3. It is thus obtained by breaking the integral defining $\Theta_n(h)$ at B and integrating over $[B,\infty)$ now the Jacobi transform of the Theta function in $\Theta_n(h)$.

12. Higher moments in the general case: Computing the moments of the reciprocal of any time–h value of $A^{(\nu)}$

$$m_k(h) = E\big[\big(A_h^{(\nu)}\big)^{-k}\big]$$

for arbitrary indices ν is not solely in terms of the Theta integrals $\Theta_k(h)$ of §11. Additional non–Theta correction terms are required. The precise result is the following

Lemma: *Let n^* be the smallest integer m such that $2m+1-|\nu-1|$ is non–negative. For any positive integer N, we then have*

$$\begin{aligned} e^{\frac{\nu^2 h}{2}} m_N(h) = {} & \sum\nolimits_{k=1}^{N} a_{N,k}\Theta_k(h) \\ & + \sum\nolimits_{n=0}^{n^*-1} \sum\nolimits_{k=0}^{N-1} \Big\{ b_{N,k}(h)\cdot C_{n,k}(h) - a_{N,k+1}\cdot D_{n,k+1}(h) \Big\}, \end{aligned}$$

for any $h > 0$.

The functions $C_{n,k}$ and $D_{n,k}$: The functions $C_{n,k}$ and $D_{n,k}$ are for any positive real number h given as follows. First,

$$C_{n,k}(h) = h^{\frac{1}{2}} \left(\frac{\beta_n h}{\sqrt{2}}\right)^{2k+1} \sum_{\ell=0}^{2k+1} (-1)^{\ell-1} \binom{2k+1}{\ell} \cdot W_\ell\left(\frac{\beta_n\sqrt{2h}}{2}\right) \cdot \left(\frac{\beta_n\sqrt{2h}}{2}\right)^{-\ell}$$

Here W_ℓ is given in terms of the complementary incomplete gamma function $\Gamma(a,x)$ by $W_\ell(x) = \exp(x^2)\cdot\Gamma((\ell+1)/2, x^2)$ for any real number x, and $\beta_n = 2n+1-|\nu-1|$. Then,

$$D_{n,k}(h) = 2^{-1}\frac{(-2\gamma_n)^k}{\left(\frac{1}{2}\right)_k}\left[\frac{1}{\sqrt{\gamma_n}} W\left(\sqrt{2h\gamma_n}\right) - \frac{\sqrt{h}}{\sqrt{\pi}}\sum_{\ell=0}^{k-1}\left(\frac{1}{2}\right)_\ell \cdot \frac{(-1)^\ell}{2\cdot(h\gamma_n)^{\ell+1}}\right]$$

where W is the function W_0, abbreviating $\gamma_n = 4^{-1}(|\nu-1|-1-2n)^2$, and with $(\lambda)_k$ the Pochhammer symbol of any complex number λ recursively given by $(\lambda)_0 = 1$ and $(\lambda)_{k+1} = (\lambda-k)\cdot(\lambda)_k$ for any non–negative integer k.

The coefficients $a_{n,k}$ and $b_{n,k}$: The coefficients $a_{n,k}$ are those of §10 Lemma. The coefficient functions $b_{n,k}(h)$ have the form

$$b_{n,n-1}(h) = b_{n,n-1,n-1}h^{-(\frac{3}{2}+2(n-1))}$$

and for any non–negative integer k less than or equal to $n-2$

$$b_{n,k}(h) = \sum\nolimits_{\ell=0}^{n-1} b_{n,k,\ell}h^{-(\frac{3}{2}+k+\ell)}.$$

They are recursively defined by

$$b_{1,0}(h) = \frac{2}{\sqrt{\pi}} h^{-\frac{3}{2}},$$

$$b_{2,0}(h) = \frac{2}{2\pi}\Big(2(1-\nu)+\frac{\nu^2}{2}\Big)h^{-\frac{3}{2}} + \frac{3}{\sqrt{2\pi}}h^{-\frac{5}{2}} \qquad \text{and} \qquad b_{2,1}(h) = -\frac{2}{\sqrt{2\pi}}h^{-\frac{7}{2}},$$

and the following coefficient recurrence rules for $n \geq 2$

$$b_{n+1,n,n} = -\frac{1}{n} b_{n,n-1,n-1}\,,$$

$$b_{n+1,n-1,\ell} = \begin{cases} 0 & \ell = 0 \\ -\frac{1}{n} b_{n,n-2,\ell-1} & \ell = 1,\dots,n-2 \\ \Big(2(n-\nu)+\frac{\nu^2}{2n}\Big) b_{n,n-1,n-1} - \frac{1}{n} b_{n,n-2,n-1} & \ell = n-1 \\ \frac{1}{n}\Big(\frac{3}{2}+2(n-1)\Big) b_{n,n-1,n-1} - \frac{1}{n} b_{n,n-2,n-1} & \ell = n. \end{cases}$$

For k any positive integer less than or equal to $n-2$ we have

$$b_{n+1,k,\ell} = \begin{cases} \Big(2(n-\nu)+\frac{\nu^2}{2n}\Big) b_{n,k,0} & \ell = 0 \\ \Big(2(n-\nu)+\frac{\nu^2}{2n}\Big) b_{n,k,\ell} + \frac{1}{n}\Big(\frac{3}{2}+k+\ell-1\Big) b_{n,k,\ell-1} - \frac{1}{n} b_{n,n-1,\ell-1} & \ell = 1,\dots,n-1 \\ \frac{1}{n}\Big(\frac{3}{2}+k+n-1\Big) b_{n,k,n-1} - \frac{1}{n} b_{n,n-1,n-1} & \ell = n, \end{cases}$$

and

$$b_{n+1,0,\ell} = \begin{cases} \Big(2(n-\nu)+\frac{\nu^2}{2n}\Big) b_{n,0,0} & \ell = 0 \\ \Big(2(n-\nu)+\frac{\nu^2}{2n}\Big) b_{n,0,\ell} + \frac{1}{n}\Big(\frac{3}{2}+\ell-1\Big) b_{n,0,\ell-1} & \ell = 1,\dots,n-1 \\ \frac{1}{n}\Big(\frac{3}{2}+n-1\Big) b_{n,0,n-1} & \ell = n. \end{cases}$$

13. An example: This section considers the example of valuing an Asian option on a non–dividend–paying stock with an interest rate $\varpi = r = 9\%$ p.a., a volatility $\sigma = 30\%$ p.a., and with time to maturity 1 year from today time 0. Suppose today's price of the stock S_0 is equal to 100, and the option has been issued today at par, i.e., $K = S_0$. Then

$\nu = 1.0$, $h = q = 0.0225$, and we have for the Theta integrals $\Theta_n(h)$ of §10:

n	$n_{c,30}$	$n_{d,30}$	$\Theta_n(h)$
1	4	4	44.2790749547
3	4	4	46822.1151330265
5	4	4	70467169.1745463509
7	4	4	116806496442.2335048967
9	4	4	202679488380050.1455046663
11	4	4	361222145817967427.8404500431
13	4	4	655202294765472642150.1383863999
15	4	4	1203316829766524026893611.5309617920
17	4	4	2230542763985591766299300393.1676737371
19	4	4	4164445349148560860975785222305.1797500863

Table 1 Growth of Theta integrals $\Theta_n(h)$.

We have computed these values using the second series of §11 with $B = 0.3$, and denote by $n_{c,30}$ and $n_{d,30}$ those indices m for which the first 30 decimal places of the partial sums $c_{B,0}+\cdots+c_{B,m}$ and $d_{B,0}+\cdots+d_{B,m}$ respectively are correct. The calculations have been done on a HP Visualize C200 using the GP/PARI CALCULATOR Version 2.0.11 (beta) of C. Batut, K. Belabas, D. Bernardi, H. Cohen and M. Olivier with 150D precision. Such a precision was desired since the coefficients $a_{n,k}$ of §10 Lemma grow like k factorial. Using this result we compute the following values of the n–th moments $m_n(h)$ of the reciprocal of Yor's accumulation process $A^{(\nu)}$ at time h:

n	$m_n(h)$
1	43.7837269185
3	91741.4898743896
5	215689033.4795106594
7	566731384819.7874280500
9	1657943864684789.3944805424
11	5380692663910949427.6422561855
13	19305805471114617878830.4144899460
15	76329904185193047892084144.9033200670
17	331513545523453183819373799801.9730752657
19	1576936784103137595901225118618858.9370371494

Table 2 Growth of the moments $m_n(h)$.

Given this rapid growth of the negative moments the question is if and when convergence of the Laguerre series is rapid enough so that only the first few of their terms are needed to reproduce the price of the Asian option. Principally, convergence of Laguerre series is effected by the oscillation of the Laguerre polynomials L_n^α in the range of positive real numbers smaller than $4n+2(\alpha+1)$. In both of our Laguerre series this is controlled by the parameter c in the sense that the smaller c the more of their terms are in their respective oscillatory range, and the better should be convergence. The Black–Scholes price of our Asian option is given as $C^{BS} = 8.83$ in [**RS**, Table 1.3] from which its normalized time–t

price $C^{(\nu)} = 0.002173850758$ is obtained on divison by 4061.916379. In the sequel we consider the ladder height density series of §8 Proposition with $\delta = -1$ and $\beta = 0$ and take $\alpha = 0$, i.e., compute with the classical Laguerre polynomials. Using the obvious notation, we record in the following examples the values the series give for these two prices after summing the first $n+1$ terms, and give the respective amounts Δ_n that have to be added to obtain their above target values. First we choose for c a "reasonable" value, to improve the series' sprint qualities so to speak.

n	$C_n^{(\nu)}$	$\Delta_n^{(\nu)}$	C_n^{BS}	Δ_n^{BS}
1	0.0041430388	0.0019691881	16.82867729	7.9986772963
3	0.0029964651	0.0008226144	12.17139078	3.3413907866
5	0.0030265291	0.0008526784	12.29350849	3.4635084905
7	0.0027607453	0.0005868946	11.21391670	2.3839167088
9	0.0024684544	0.0002946036	10.02665531	1.1966553198
11	0.0023864317	0.0002125809	9.69348587	0.8634858736
13	0.0023930119	0.0002191611	9.72021406	0.8902140555
15	0.0023544102	0.0001805595	9.56341738	0.7334173791
17	0.0022627357	0.0000888849	9.19104301	0.3610430106
19	0.0021738508	4.049×10^{-15}	8.83000000	1.645×10^{-11}

Table 3 Ladder height series with $c = 1.367054258545$

As to be expected for this magnitude of c, there is a steady convergence of the series. One should expect it to converge mildly oscillating. However, mind the following fallacy. It is not sufficient to have the the normalized time–t price $C^{(\nu)}$ correct up to the the first two or three decimal places only! Indeed, we have this accuracy for $C^{(\nu)}$ from the outset while there are still big deviations from the Black–Scholes price C^{BS}. These phenomenons are much more pointed if a larger value for c is chosen. Indeed notice how the following series actually start with almost the correct values, then wildly oscillate, and only begin to calm down after fifteen terms or so.

n	$C_n^{(\nu)}$	$\Delta_n^{(\nu)}$	C_n^{BS}	Δ_n^{BS}
1	0.0020571929	−0.0001166579	8.35614534	−0.473854662
3	0.0045737040	0.0023998531	18.57800280	9.748002806
5	0.0044363733	0.0022625225	18.02017734	9.190177342
7	−0.0055198391	−0.0076936899	−22.42112502	−31.251125020
9	0.0028067490	0.0006328989	11.40077951	2.570779513
11	0.0034914141	0.0013175634	14.18183220	5.351832204
13	0.0017503210	−0.0004235298	7.10965752	−1.720342482
15	0.0015816389	−0.0005922118	6.42448511	−2.405514894
17	0.0017276113	−0.0004462395	7.01741241	−1.812587586
19	0.0017603640	−0.0004134868	7.15045128	−1.679548715

Table 4 Ladder height series with $c = 6$

In summary, we are now able to efficiently compute the negative moments of Yor's accumulation processes that enter into the Laguerre series for the Asian option. However, there has still work to be done in characterizing optimal choices of the convergence parameter c.

14. Epilogue: In this paper we have concentrated on the first of the two ladder height density Laguerre series of §8 Proposition. Working with the second of these requires to have for certain positive real numbers α, β at least computable expressions for the expectations

$$f_{\alpha,\beta}(t) = E^Q\left[X^{-\alpha}e^{-\frac{\beta}{2X}}\right]$$

for any time $t > 0$ setting $X = A_t^{(\nu)}$. Adapting the argument of [**Y**, §3, p.515f], their Laplace transform with respect to time is computed in terms of modified Bessel functions I_μ and MacDonald functions K_μ as follows

$$\mathcal{L}(f_{\alpha,\beta})(z) = \frac{2}{\eta^{\alpha-\nu}}\int_0^\infty \xi^{\alpha-1} I_{\sqrt{2z}}(\xi) K_{\alpha-\nu}(\eta\xi)\, d\xi$$

setting $\eta = ((1+\beta)/2)^{1/2}$ and with z any complex number whose real part is positive and such that $\mathrm{Re}\,((2z)^{1/2}) > 1-\nu$. In analogy to the modified Weber–Schafheitlin integrals of [**W**, p.410] these Laplace transforms can be expressed using the Gauss hypergeometric function ${}_2F_1$. On Laplace inversion $f_{\alpha,\beta}$ is seen to be given by the contour integral

$$f_{\alpha,\beta}(t) = \frac{2}{\eta^{\alpha-\nu}}\frac{1}{2\pi i}\int_{\log C_R} \frac{w}{\sqrt{2\pi t^3}} e^{-\frac{w^2}{2t}} \int_0^\infty \xi^{\alpha-1} K_{\alpha-\nu}(\eta\xi) e^{\xi \cosh(w)} d\xi\, dw\,.$$

Here $\log C_R$ is the following path of integration

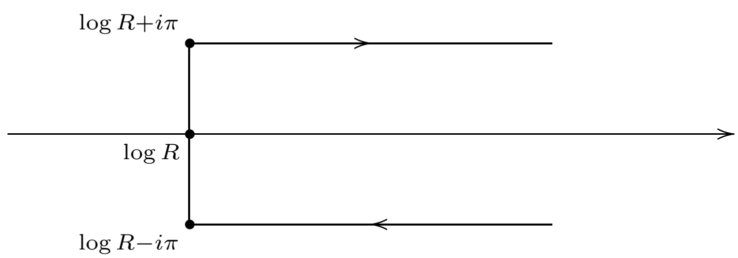

Figure 1 The contour $\log C_R$

where the positive real number R has to be chosen so big that the real parts of w^2 and w of any of its elements w are positive and such that η is bigger than $\cosh(\log R)$. Under additional hypotheses, from [**E**, 7.7.3 (26), p.50] the inner integral can be expressed using the Gauss hypergeometric function ${}_2F_1$. Alternative angles on these expectations, like their representation as Kantorovich–Lebedev transforms, are discussed in [**AG**].

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December 2000